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Yoo, Jaiyul ; Gong, Jinn-Ouk

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Exact analytic solution for non-linear density fluctuation in a Λ CDM universe

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Abstract

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1 Introduction

The coming decades will witness the golden age in cosmology with large scale galaxy surveys, as numerous ambitious programs such as Euclid, WFIRST, LSST and so on, will be in full operation, measuring tens and hundreds of millions of galaxies in the sky and delivering an unprecedented amount of data with unprecedented precision. Taking full advantage of these impressive experimental and observational developments requires substantial advances in theoretical modeling. In this regard, the recent development of the relativistic description of galaxy clustering [1, 2] calls for more endeavour in theoretical description on large scales, where the relativistic effects in galaxy clustering become important but has been ignored in the standard treatment of galaxy clustering due to the lack of theoretical understanding and the large measurement uncertainties.

The relativistic effects are intrinsically present in galaxy clustering, since all the galaxy clustering observables are obtained by measuring light from the source galaxies and the light propagation is subject to the same relativistic effects that we measure in the cosmic microwave background (CMB). One of the well-known relativistic effects in CMB is the Sachs-Wolfe effect, with which photons lose their energy, climbing out of the gravitational potential [3]. The same relativistic effect changes the observed redshift of galaxies we measure in galaxy surveys, as it changes the temperature of CMB photons. Another important example of the relativistic effects in galaxy clustering is the three-dimensional volume distortion in four-dimensional spacetime mapped by the observed redshift and angular positions. A complete treatment of all the effects in galaxy clustering was given in [4], clarifying the effects that involve the intrinsic properties of the source galaxies (“source” effects) and those that involve the change of the volume the surveys cover (“volume” effects). Such relativistic effects in galaxy clustering were previously unaccounted for in the standard method. The full relativistic description of galaxy clustering was developed in [1, 2, 5], providing new opportunities to probe cosmology through subtle but unique relativistic effects: see also [6, 7, 8] for different derivations, and see [9] for review.

Going forward beyond linear theory, the second-order relativistic description of galaxy clustering has been recently formulated [10, 11, 12] to extract additional information from the higher-order statistics such as the bispectrum. More work needs to be done for the complete description, and in particular one of the critical elements in generalizing the formalism beyond linear order is galaxy bias that relates the galaxy number density to the underlying matter distribution. While it has been extensively studied in the Newtonian framework, generalizing it to the relativistic framework requires more work — galaxy biasing was left out in [12], and the proper-time hypersurface was advocated in [10, 11].

The proper-time hypersurface of nonrelativistic matter flows is a physically well-defined 3-hypersurface a local observer can establish, who is moving together with nonrelativistic matter flows such as dark matter or baryons on large scales. This can be described by any choice of gauge conditions, but the comoving gauge choice in a universe with a pressureless medium allows the global coordinate system to be aligned to the proper-time hypersurface, facilitating the computation of the matter density fluctuation in the proper-time hypersurface [13]. This aspect is of particular importance when galaxy bias is considered. Beyond the linear order in perturbations, the spatial gauge conditions make a difference in physical quantities, even with the same temporal gauge condition (comoving gauge in our case). In [11], the synchronous comoving gauge was advocated for galaxy bias. However, it was shown [13, 14] that the spatial coordinates in this gauge condition trace the nonrelativistic matter flows and the direct computation of the matter power spectrum in this coordinate is inadequate for galaxy bias. In contrast, the comoving gauge condition with the spatial C-gauge condition fixes the spatial coordinates, providing a natural framework to describe local dynamics in the relativistic context [13]. See Section 2 for the detail of the gauge conditions mentioned above.

In this work, we derive the exact third-order analytic solution of the matter density and the velocity fluctuations in a Λ CDM universe, substantially extending the works in the Einstein-de Sitter (EdS) universe [15, 16]. The matter density fluctuation is the dominant contribution to galaxy clustering on all scales, and there is no relativistic correction to it at the linear order in perturbation. Compared to the leading-order power spectrum, the subtle relativistic corrections that contain additional information require the third-order relativistic calculation. Our work greatly expands the calculations in [15, 16], accounting for the explicit time-dependence of each contribution, providing extensive studies of nonlinear relativistic equations, and clarifying the difference in our solution to the previous works [14, 17, 18, 19].

The organization of this article is as follows. In Section 2 we derive the nonlinear dynamical equations for the density and the velocity fluctuations in the proper-time hypersurface of nonrelativistic matter flows. Full third-order analytic solutions are presented in Section 3. Our analytic solutions are then compared to the work in [18, 19] in Section 4, and they are casted in terms of standard Fourier kernels in comparison to the standard Newtonian perturbation theory in Section 5. Finally, we end in Section 6 with a discussion of further implication. Throughout the article we will use a, b, c, \dots to represent the spacetime indices and i, j, k, \dots to represent the spatial indices. We assume a flat space with the Friedmann-Robertson-Walker (FRW) metric.

2 Nonlinear dynamical equations

Here we briefly review the Arnowitt-Deser-Misner (ADM) formalism to describe the nonlinear dynamics and present our notation convention for the spacetime metric and its perturbations.

Given the spacetime metric g_{ab} and its coordinate system x^a , the ADM formalism considers spatial hypersurfaces labeled by its time coordinate t . The induced spatial metric $h_{ij} = g_{ij}$ of these 3-hypersurfaces is treated as the dynamical degrees of freedom, subject to the constraint equations. The spacetime metric in the ADM formalism is conventionally written as [20, 21]

$$ds^2 = g_{ab}dx^a dx^b = (-N^2 + N^i N_i) dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j, \quad (2.1)$$

where the lapse function N represents the change in the proper time between two spatial hypersurfaces with Δt , the shift vector N^i represents the change in the normal direction n^a of the hypersurface, and the spatial indices are lowered by the spatial metric h_{ij} (e.g., $N_i = h_{ij} N^j$). The normal vector (or often called the *normal observer*) to the 3-hypersurface is

$$n^a = \left(\frac{1}{N}, -\frac{1}{N} N^i \right), \quad n_a = g_{ab} n^b = (-N, 0), \quad n_a n^a = -1. \quad (2.2)$$

Once the energy-momentum tensor T_{ab} is specified, the ADM fluid quantities can be derived, representing the energy density, the momentum density, and the stress tensor measured by the normal observer:

$$E = n_a n_b T^{ab} = N^2 T^{00}, \quad J_i = -n_a T_i^a = N T_i^0, \quad S_{ij} = T_{ij}. \quad (2.3)$$

In addition, the extrinsic curvature tensor K_{ij} describes the local bending of 3-hypersurfaces embedded in the four-dimensional spacetime:

$$K_{ij} = \frac{1}{2N} \left(N_{i;j} + N_{j;i} - \dot{h}_{ij} \right), \quad K \equiv h^{ij} K_{ij}, \quad \bar{K}_{ij} \equiv K_{ij} - \frac{1}{3} h_{ij} K, \quad (2.4)$$

We now make connections to a FRW universe, where the metric is described by the usual FRW metric and small perturbations around the background. Given the metric convention, the general relations between the ADM variables and the metric perturbations can be derived [22, 23]. However, since we are interested in the proper-time hypersurface of nonrelativistic matter flows, we first impose a gauge condition, greatly simplifying the manipulation.

We choose the *comoving gauge* as our temporal gauge condition, in which a local observer with the four velocity u^a moves along the flow of nonrelativistic matter and sees vanishing energy flux in the observer rest frame. Furthermore, the comoving gauge condition $T^0_i = 0$ is greatly simplified, if we consider a universe composed of nonrelativistic matter only: as the energy-momentum tensor in this case is $T_{ab} = \rho_m u_a u_b$ with the matter density ρ_m , the comoving gauge condition becomes $u_i = 0$, aligning the local observer u^a with the normal observer n^a , with which the ADM fluid description is directly applicable to the physical system of interest. The observer four velocity is then decomposed in terms of the shear σ_{ij} and the expansion θ [24, 25] as

$$u_{a;b} = n_{a;b} = \frac{1}{3}\theta h_{ab} + \sigma_{ab} , \quad \theta = n^a{}_{;a} = -K , \quad \sigma_{ij} = n_{(i;j)} = -\bar{K}_{ij} , \quad (2.5)$$

where the semicolon is the covariant derivative with respect to the spacetime metric g_{ab} and the induced metric $h_{ab} = g_{ab} + u_a u_b$ is indeed the projection to the 3-hypersurface. The normal observer is irrotational $u_{[a;b]} = 0$, and the energy-momentum conservation of the nonrelativistic matter flows imposes that the observer follows the geodesic $a_a = u_{a;b} u^b = 0$ and $N = 1$ [13]. Thus the coordinate time *exactly* corresponds to the proper time.

In addition, as our spatial gauge condition we choose the C-*gauge* [22], such that the spacetime metric takes the form such that the off-diagonal term in the spatial metric g_{ij} is removed:

$$g_{00} = -1 + N^i N_i \equiv -1 - 2\alpha , \quad g_{0i} = N_i \equiv -\nabla_i \chi , \quad g_{ij} = h_{ij} \equiv a^2(1 + 2\varphi)\bar{g}_{ij} , \quad (2.6)$$

where the spatial gradient ∇^i is based on the background 3-metric \bar{g}_{ij} . It is noted [22] that the C-gauge condition leaves no residual gauge freedom when combined with our temporal gauge condition. We assume no vector or tensor perturbations in the spacetime metric.

In contrast, as the spatial gauge condition one can opt to choose the B-*gauge* [22], in addition to the same temporal comoving gauge condition. This choice is often called the *comoving-synchronous* gauge, and the metric becomes

$$g_{00} = -1 , \quad g_{0i} \equiv N_i = 0 , \quad g_{ij} = h_{ij} = a^2 [(1 + 2\varphi)\bar{g}_{ij} + 2\nabla_i \nabla_j \gamma] . \quad (2.7)$$

While this choice also corresponds to the proper-time hypersurface, the spatial coordinates are changing in time, tracing the nonrelativistic matter flows in a way similar to the Lagrangian coordinates in the Newtonian dynamics and leaving flows at rest in a given spatial coordinate. In this work, no further investigation is made along this direction.

In our comoving C-gauge condition, the local observer u^a moving with the nonrelativistic matter flows becomes the normal observer n^a , facilitating the use of the ADM formalism in a physically meaningful way. The ADM quantities in our case are greatly simplified as

$$N = 1 , \quad E = T^{00} = \rho_m , \quad J_i = 0 , \quad S_{ij} = \bar{S}_{ij} = S = 0 , \quad (2.8)$$

where the scalar part S and the traceless part \bar{S}_{ij} of the stress tensor S_{ij} are defined in the same manner as those of the extrinsic curvature K_{ij} in (2.4).

The relevant nonlinear equations based on the ADM variables are the conservation and constraint equations of energy and momentum, and the trace and tracefree parts of the dynamical equations. The complete set of the ADM equations can be found in [20, 26, 27] and we do not present them here. At the background level in perturbations, the nonlinear equations correspond to the familiar matter density conservation and the Friedmann equations. With the background evolution removed, the nonlinear dynamical equations yield a series of nonlinear perturbation equations to be solved for the density fluctuation $\delta \equiv \rho_m / \bar{\rho}_m - 1$, with $\bar{\rho}_m$ being the background density, and the perturbation in the extrinsic curvature $\kappa \equiv 3H + K$. The master dynamical equations are the conservation equation and the Raychaudhuri equation:

$$\dot{\delta} - \kappa = N^i \nabla_i \delta + \delta \kappa , \quad (2.9)$$

$$\dot{\kappa} + 2H\kappa - 4\pi G\bar{\rho}_m \delta = N^i \nabla_i \kappa + \frac{1}{3}\kappa^2 + \sigma^{ij}\sigma_{ij} , \quad (2.10)$$

supplemented by the constraint equations:

$$\delta R = \sigma^{ij}\sigma_{ij} + 4H\kappa - \frac{2}{3}\kappa^2 + 16\pi G\bar{\rho}_m\delta , \quad (2.11)$$

$$\frac{2}{3}\nabla_i\kappa = \sigma^j_{i;j} . \quad (2.12)$$

In order to solve these dynamical equations perturbatively, we need to compute the nonlinear perturbation variables at each order such as the metric perturbations and the geometric quantities of 3-hypersurfaces. For example, up to the third order in perturbations, N^i , κ and σ_{ij} can be written by using (2.4) and (2.5) as

$$N^i = -\frac{1}{a^2}\nabla^i\chi(1-2\varphi+4\varphi^2) , \quad (2.13)$$

$$\kappa = -3\dot{\varphi} - \frac{\Delta}{a^2}\chi + 6\varphi\dot{\varphi} + \frac{1}{a^2}[2\varphi\Delta\chi(1-2\varphi) - \nabla_i\chi\nabla^i\varphi(1-4\varphi)] , \quad (2.14)$$

$$\sigma_{ij} = \left(\nabla_i\nabla_j - \frac{1}{3}\delta_{ij}\Delta\right)\chi - 2\left(\nabla_{(i}\varphi\nabla_{j)}\chi - \frac{1}{3}\delta_{ij}\nabla^k\varphi\nabla_k\chi\right)(1-2\varphi) . \quad (2.15)$$

The momentum constraint equation can be arranged as

$$\begin{aligned} \kappa + \frac{1}{a^2}\Delta\chi &= \frac{1}{a^2}[2\varphi\Delta\chi(1-2\varphi) - \nabla^i\varphi\nabla_i\chi(1-4\varphi)] \\ &+ \frac{3}{2a^2}\Delta^{-1}\nabla^i\left[\nabla_i\chi\Delta\varphi + \nabla_j\nabla_i\varphi\nabla^j\chi - 4\varphi(\nabla^j\chi\nabla_j\nabla_i\varphi + \Delta\varphi\nabla_i\chi) - (\nabla_i\chi\nabla_j\varphi + 3\nabla_j\chi\nabla_i\varphi)\nabla^j\varphi\right] . \end{aligned} \quad (2.16)$$

We ignored the vector and the tensor contributions (but see [28]). Furthermore, combining the definition of κ in (2.14) with the ADM momentum constraint, we derive the dynamical equation for the curvature potential:

$$\dot{\varphi} = 2\varphi\dot{\varphi} - \frac{1}{2a^2}\Delta^{-1}\nabla^i\left[\nabla^j\chi\nabla_j\nabla_i\varphi + \Delta\varphi\nabla_i\chi - 4\varphi(\nabla^j\chi\nabla_j\nabla_i\varphi + \Delta\varphi\nabla_i\chi) - (\nabla_i\chi\nabla_j\varphi + 3\nabla_j\chi\nabla_i\varphi)\nabla^j\varphi\right] . \quad (2.17)$$

At the linear order in perturbations, the curvature potential is a time-independent spatial function set by the initial condition $\varphi^{(1)} \equiv \mathcal{R}(\mathbf{x})$. This remains true to all orders in perturbation on super-horizon scales, where the gradient terms are negligible. On sub-horizon scales, the curvature potential evolves in time beyond the linear order in perturbations, and we need to evaluate the time-dependence of the nonlinear terms in (2.17) before we integrate to obtain the time-evolution of the curvature potential.

3 Analytic solutions of the matter density fluctuation

Armed with the nonlinear equations in Section 2, in this section we now derive the third-order analytic solution of the matter density fluctuation δ in a Λ CDM universe.

3.1 Battle plan

The analytic derivation of the third-order solutions in general relativity inevitably involves many steps technical and lengthy in nature, so we start by presenting the master differential equation and the overall strategy to solve the differential equation at each order in perturbations.

Using the continuity equation (2.9), the ADM energy constraint (2.11) can be rearranged as the master differential equation for δ :

$$\mathcal{H}\delta' + \frac{3}{2}\mathcal{H}^2\Omega_m\delta = \frac{a^2}{4}\left(\delta R - \sigma^{ij}\sigma_{ij} + \frac{2}{3}\kappa^2 + 4HN^i\nabla_i\delta + 4H\delta\kappa\right) , \quad (3.1)$$

where the prime is the derivative with respect to the conformal time $d\eta = dt/a$, $\mathcal{H} = a'/a = aH$ is the conformal Hubble parameter and $\Omega_m = 8\pi G\bar{\rho}_m/(3H^2)$. The left-hand side (LHS) of (3.1) is linear in δ , and the right-hand

side (RHS) is composed of at least quadratic terms, except the intrinsic curvature of 3-hypersurface δR , such that n -th order solution $\delta^{(n)}$ can be used to compute $(n+1)$ -th order terms in $\text{RHS}^{(n+1)}$ and derive $(n+1)$ -th order solution $\delta^{(n+1)}$ in the LHS. Once n -th order solution $\delta^{(n)}$ is derived, the solution $\kappa^{(n)}$ can be obtained algebraically by using the ADM energy constraint, explicitly written as

$$\begin{aligned} \frac{3}{2}H^2\Omega_m\delta + H\kappa + \frac{1}{a^2}\Delta\varphi &= \frac{1}{6}\kappa^2 + \frac{1}{12a^4}[(\Delta\chi)^2 - 3\nabla_i\nabla_j\chi\nabla^i\nabla^j\chi](1-4\varphi) + \frac{1}{a^2}\left(4\varphi\Delta\varphi + \frac{3}{2}\nabla^i\varphi\nabla_i\varphi\right) \\ &+ \frac{1}{a^4}\left(\nabla^j\nabla^i\chi\nabla_j\varphi\nabla_i\chi - \frac{1}{3}\nabla^i\varphi\nabla_i\chi\Delta\chi\right) - \frac{3}{a^2}\varphi(3\nabla^i\varphi\nabla_i\varphi + 4\varphi\Delta\varphi) . \end{aligned} \quad (3.2)$$

The homogeneous solution that satisfies (3.1) with vanishing RHS is readily derived as $\delta_h \propto H$ and is identified as the usual decaying mode in the standard Newtonian solution, which we ignore henceforth. The particular solution with nonvanishing RHS corresponds to the growing mode solution:

$$\delta_p = \delta_h \int \frac{d\tau}{\delta_h} \left(\frac{\text{RHS}}{\mathcal{H}} \right) = H \int dt \left(\frac{\text{RHS}}{\mathcal{H}^2} \right) , \quad (3.3)$$

where RHS is of dimension two. To compute RHS of (3.1) at each order in perturbations, we split RHS as the sum of perturbative expansion terms:

$$\text{RHS} \equiv \sum_n \text{RHS}^{(n)} = \text{RHS}^{(1)} + \text{RHS}^{(2)} + \text{RHS}^{(3)} + \dots , \quad (3.4)$$

where the superscripts represent the order of each term in perturbative expansions. First, we write $\text{RHS}^{(n)}$ at each perturbation order as the sum of terms $\text{RHS}_m^{(n)}$:

$$\text{RHS}^{(n)}(t, \mathbf{x}) \equiv \sum_{m=1}^n \text{RHS}_m^{(n)}(t, \mathbf{x}) , \quad (3.5)$$

where the subscript m indicates that the time-dependence in the EdS universe scales as $\text{RHS}_m^{(n)}(t, \mathbf{x}) \propto D_1^m(t)$. Each of these $\text{RHS}_m^{(n)}$ is decomposed as the sum of the scale-dependent and time-dependent functions:

$$\text{RHS}_m^{(n)}(t, \mathbf{x}) = \sum_{I=A,B,\dots} X_{mI}^{(n)}(\mathbf{x}) T_{mI}(t) , \quad (3.6)$$

where the subscript I denotes different time dependences that become identical as $T_{mI}(t) \propto D_1^m(t)$ in the EdS universe, and $X_{mI}^{(n)}(\mathbf{x})$ is a time-independent but scale-dependent function at n -th order in perturbations.

According to this decomposition, the growing mode solution will be the sum of individual solutions $\delta_{mI}^{(n)}$ with corresponding $\text{RHS}_{mI}^{(n)}$:

$$\delta_{mI}^{(n)}(t, \mathbf{x}) = H \int dt \left(\frac{\text{RHS}_{mI}^{(n)}}{\mathcal{H}^2} \right) = D_{mI}(t) X_{mI}^{(n)}(\mathbf{x}) \quad \text{with} \quad D_{mI}(t) = H \int dt \frac{T_{mI}(t)}{\mathcal{H}^2} , \quad (3.7)$$

where $D_{mI}(t)$ is of dimension minus two. It is noted that $\delta_{mI}^{(n)}$ is at n -th order in perturbations and its time-dependence $D_{mI}(t)$ is determined by the time-dependent function $T_{mI}(t)$ in $\text{RHS}_{mI}^{(n)}$. Therefore, the full solution is then

$$\delta_p = \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + \dots \quad \text{where} \quad \delta^{(n)}(t, \mathbf{x}) = \sum_{m,I} \delta_{mI}^{(n)}(t, \mathbf{x}) . \quad (3.8)$$

The main result of this section is this analytic solution up to third order, given by (3.14), (3.23) and (3.46).

A further manipulation can be made to facilitate the computation by defining the logarithmic growth rate $f_{mI}(t)$ associated with $D_{mI}(t)$:

$$f_{mI}(t) \equiv \frac{d \ln D_{mI}(t)}{d \ln a} , \quad D_{mI}' = \mathcal{H} f_{mI} D_{mI} . \quad (3.9)$$

Using the logarithmic growth rate, the LHS of (3.1) can be written as

$$\text{LHS} \left[\delta_{mI}^{(i)} \right] = \mathcal{H} \delta_{mI}^{(i)'} + \frac{3}{2} \mathcal{H}^2 \Omega_m \delta_{mI}^{(i)} \equiv \mathcal{H}^2 f_{mI} \Sigma_{mI} \delta_{mI}^{(i)}, \quad \Sigma_{mI}(t) = 1 + \frac{3}{2} \frac{\Omega_m}{f_{mI}}, \quad (3.10)$$

and the growth rate is then related to the logarithmic growth rate as

$$D_{mI}(t) = \frac{T_{mI}}{\mathcal{H}^2 f_{mI} \Sigma_{mI}}, \quad f_{mI}(t) = \frac{D'_{mI}}{\mathcal{H} D_{mI}} = -\frac{3}{2} \Omega_m + \frac{T_{mI}}{\mathcal{H}^2 D_{mI}} = \frac{T_{mI}}{\mathcal{H}^2 \Sigma_{mI} D_{mI}}, \quad (3.11)$$

providing a convenient way of computing the logarithmic growth rate without taking numerical differentiation of the growth factor.

3.2 Linear- and second-order solutions

We start by deriving the well-known linear- and second-order solutions to provide the guidance of the strategy laid in Section 3.1. At the linear order in perturbations, the RHS of (3.1) is simply

$$\text{RHS}^{(1)}(\mathbf{x}) = -\Delta \varphi^{(1)}(\mathbf{x}) \equiv -\Delta \mathcal{R}(\mathbf{x}) = X_1^{(1)}(\mathbf{x}) \quad \text{with} \quad T_1(t) = 1, \quad (3.12)$$

so that the linear-order growth solution is then

$$D_1(t) = H \int \frac{dt}{\mathcal{H}^2} = \frac{1}{\mathcal{H}^2 f_1 \Sigma_1}, \quad (3.13)$$

$$\delta_1^{(1)}(t, \mathbf{x}) = D_1(t) X_1^{(1)}(\mathbf{x}) = -\frac{\Delta \mathcal{R}(\mathbf{x})}{\mathcal{H}^2 f_1 \Sigma_1}, \quad (3.14)$$

where the linear-order growth factor D_1 needs to be numerically integrated before the logarithmic growth rate f_1 is obtained.¹ The linear growth factor D_1 is identical to one in the standard Newtonian description, once normalized to remove its dimension at some epoch (see Section 5.2). According to the ADM energy constraint (3.2) and the ADM momentum constraint (2.16), we can derive the linear-order perturbation to the extrinsic curvature $\kappa_1^{(1)2}$ and the scalar shear $\chi^{(1)}$ as

$$\frac{\kappa_1^{(1)}(t, \mathbf{x})}{H f_1} = \delta_1^{(1)}(t, \mathbf{x}) = -\frac{\Delta \mathcal{R}(\mathbf{x})}{\mathcal{H}^2 f_1 \Sigma_1} \quad \text{and} \quad \chi^{(1)}(t, \mathbf{x}) = -a^2 \Delta \kappa^{(1)} = \frac{\mathcal{R}(\mathbf{x})}{H \Sigma_1}. \quad (3.15)$$

To compute the RHS of (3.1) to the second order in perturbations, first we need to derive the second-order curvature potential by analytically integrating (2.17) over time:

$$\varphi^{(2)}(t, \mathbf{x}) = \mathcal{R}^{(2)}(\mathbf{x}) - \frac{1}{2\mathcal{H}^2 f_1 \Sigma_1} \left[\frac{1}{2} \nabla^i \mathcal{R} \nabla_i \mathcal{R} + \Delta^{-1} \nabla^i (\nabla_i \mathcal{R} \Delta \mathcal{R}) \right] \equiv \mathcal{R}^{(2)} + \varphi_2^{(2)}, \quad (3.16)$$

where $\mathcal{R}^{(2)}(\mathbf{x})$ is the initial condition at the second order and the quadratic terms are evaluated at the linear order in perturbations. The curvature potential grows in time at the second order in proportion to the growth factor $D_1(t)$, but they still vanish on superhorizon scales.

Following the strategy in Section 3.1, the RHS of (3.1) at the second order in perturbations is written as

$$\begin{aligned} \text{RHS}^{(2)} = & -\Delta \mathcal{R}^{(2)} + \frac{3}{2} \nabla^i \mathcal{R} \nabla_i \mathcal{R} + 4\mathcal{R} \Delta \mathcal{R} + \frac{1}{\mathcal{H}^2 f_1 \Sigma_1} \frac{\Delta}{2} \left[\frac{1}{2} \nabla^i \mathcal{R} \nabla_i \mathcal{R} + \Delta^{-1} \nabla^i (\nabla_i \mathcal{R} \Delta \mathcal{R}) \right] \\ & + \frac{1}{\mathcal{H}^2 \Sigma_1^2} \frac{1}{4} \left[(\Delta \mathcal{R})^2 - \nabla^i \nabla^j \mathcal{R} \nabla_i \nabla_j \mathcal{R} \right] + \frac{1}{\mathcal{H}^2 f_1 \Sigma_1^2} \left[(\Delta \mathcal{R})^2 + \nabla^i \mathcal{R} \Delta \nabla_i \mathcal{R} \right]. \end{aligned} \quad (3.17)$$

¹In a Λ CDM universe, the linear-order growth factor can be analytically computed in terms of the associated Legendre function of the second kind [29], while it still needs to be numerically evaluated.

²Inspecting the time-dependence of the continuity equation (2.9), we find this relation remains valid to all orders in perturbation, i.e. for the constant terms on the RHS of (3.1) irrespective of perturbation order,

$$\frac{\kappa_1(t, \mathbf{x})}{H f_1} = \delta_1(t, \mathbf{x}) = \frac{\text{RHS}_1(\mathbf{x})}{\mathcal{H}^2 f_1 \Sigma_1},$$

with $\text{RHS}_1 = \sum_n \text{RHS}_1^{(n)}$ [see (3.19) and (3.28)].

So there are four different time dependences, including $T_1 = 1$ for the first 3 terms on the RHS of (3.17) that leads to the linear-order growth factor $D_1(t)$ given by (3.13). Thus, other than D_1 , we find the new second-order growth factors for $\delta^{(2)}$ as

$$D_{2A} = \frac{7}{5}H \int dt D_1^2 f_1 \Sigma_1 = \frac{D_1^2 f_1 \Sigma_1}{f_{2A} \Sigma_{2A}}, \quad D_{2B} = \frac{7}{2}H \int dt D_1^2 f_1^2 = \frac{D_1^2 f_1^2}{f_{2B} \Sigma_{2B}}, \quad D_{2C} = \frac{7}{2}H \int dt D_1^2 f_1 = \frac{D_1^2 f_1}{f_{2C} \Sigma_{2C}}, \quad (3.18)$$

with the corresponding time-independent spatial functions

$$X_1^{(2)} = -\Delta \mathcal{R}^{(2)} + \frac{3}{2} \nabla^i \mathcal{R} \nabla_i \mathcal{R} + 4\mathcal{R} \Delta \mathcal{R} \quad (3.19)$$

for $D_1(t)$ and for $D_{2I}(t)$

$$X_{2A}^{(2)} = \frac{5}{14} \left[\nabla_i (\nabla^i \mathcal{R} \Delta \mathcal{R}) + \frac{\Delta}{2} (\nabla^i \mathcal{R} \nabla_i \mathcal{R}) \right], \quad (3.20)$$

$$X_{2B}^{(2)} = \frac{1}{14} \left[\nabla_i (\nabla^i \mathcal{R} \Delta \mathcal{R}) - \frac{\Delta}{2} (\nabla^i \mathcal{R} \nabla_i \mathcal{R}) \right], \quad (3.21)$$

$$X_{2C}^{(2)} = \frac{2}{7} \nabla_i (\nabla^i \mathcal{R} \Delta \mathcal{R}). \quad (3.22)$$

Thus, the total second-order solution associated with RHS⁽²⁾ is

$$\delta^{(2)}(t, \mathbf{x}) = \delta_1^{(2)} + \sum_{I=A}^C \delta_{2I}^{(2)} = D_1 X_1^{(2)} + \sum_{I=A}^C D_{2I} X_{2I}^{(2)}. \quad (3.23)$$

Note that not all D_{2I} 's are independent but they are subject to the constraint $D_{2A} + D_{2C} = 2D_1^2$. This allows us to rearrange $\delta_2^{(2)} \equiv \sum_I \delta_{2I}^{(2)}$ the same as the standard Newtonian form:

$$\delta_2^{(2)}(t, \mathbf{x}) = \frac{5D_{2A} + D_{2B} + 4D_{2C}}{10} \left[\frac{5}{7} \nabla_i (\nabla^i \mathcal{R} \Delta \mathcal{R}) \right] + \frac{5D_{2A} - D_{2B}}{4} \left[\frac{\Delta}{7} (\nabla^i \mathcal{R} \nabla_i \mathcal{R}) \right]. \quad (3.24)$$

Note that the two pure spatial functions inside the square brackets exactly correspond to the Newtonian second-order kernels $A_2(\mathbf{k})$ and $B_2(\mathbf{k})$ in the Fourier space: see (5.11).

The second-order extrinsic curvature perturbation $\kappa^{(2)}$ and the scalar shear $\chi^{(2)}$ can be computed from the ADM energy constraint (3.2) and the momentum constraint equation (2.16) respectively, resulting

$$\begin{aligned} \kappa^{(2)} &= \dot{\delta}_1^{(2)} + \frac{1}{2} \frac{d}{dt} \left\{ 2 \sum_{I=A}^C \delta_{2I}^{(2)} - D_1^2 [\nabla^i \mathcal{R} \nabla_i \Delta \mathcal{R} + (\Delta \mathcal{R})^2] \right\} \equiv \dot{\delta}_1^{(2)} + \frac{\dot{\mathcal{K}}}{2}, \\ \chi^{(2)} &= \frac{1}{H \Sigma_1} \left[\mathcal{R}^{(2)} - \mathcal{R}^2 - \frac{1}{2} \nabla^i \mathcal{R} \nabla_i \mathcal{R} + \frac{3}{2} \Delta^{-2} \nabla_i \nabla_j (\nabla^i \mathcal{R} \nabla^j \mathcal{R}) \right] - a^2 \Delta^{-1} \kappa_2^{(2)} \equiv \chi_1^{(2)} - \frac{a^2}{2} \Delta^{-1} \dot{\mathcal{K}}. \end{aligned} \quad (3.25)$$

Note from above that $H \Sigma_1 \chi_1^{(2)}$ constant, and that \mathcal{K} contains four different time dependences: D_{2A} , D_{2B} , D_{2C} and D_1^2 , which all become identical to D_1^2 in the EdS universe.

3.3 Third-order solutions

At the third order in perturbations, we need to consider cubic terms, consisting of three perturbation variables evaluated at the linear order to make the cubic term at the third order. In addition, we need to consider quadratic terms that were evaluated in the previous section at the second order, because those quadratic terms also contribute to the third order with one variable at the second order and the other at the linear order.

Following the same strategy in Section 3.2, we first integrate (2.17) to derive the third-order curvature

potential. We then find

$$\begin{aligned}\varphi^{(3)}(t, \mathbf{x}) = & \mathcal{R}^{(3)}(\mathbf{x}) + 2\mathcal{R}\mathcal{R}^{(2)} + 2\mathcal{R}\varphi_2^{(2)} + 2D_1\Delta^{-1}\nabla_i \left(\nabla^i \mathcal{R} \nabla^j \mathcal{R} \nabla_j \mathcal{R} + \mathcal{R} \nabla^i \nabla^j \mathcal{R} \nabla_j \mathcal{R} + \mathcal{R} \Delta \mathcal{R} \nabla^i \mathcal{R} \right) \\ & - \frac{D_1}{2} \Delta^{-1} \nabla_i \left(\nabla^i \nabla^j \mathcal{R}^{(2)} \nabla_j \mathcal{R} + \Delta \mathcal{R}^{(2)} \nabla^i \mathcal{R} \right) - \frac{D_1}{4} \Delta^{-1} \nabla_i \left(\nabla^i \nabla^j \varphi_2^{(2)} \nabla_j \mathcal{R} + \Delta \varphi_2^{(2)} \nabla^i \mathcal{R} \right) \\ & - \frac{D_1}{2} \Delta^{-1} \nabla_i \left[\nabla^i \nabla^j \mathcal{R} \nabla_j \left(H \Sigma_1 \chi_1^{(2)} \right) + \Delta \mathcal{R} \nabla^i \left(H \Sigma_1 \chi_1^{(2)} \right) \right] + \frac{\Delta^{-1}}{4} \nabla_i \left(\nabla^i \nabla^j \mathcal{R} \Delta^{-1} \nabla_j \mathcal{K} + \Delta \mathcal{R} \Delta^{-1} \nabla^i \mathcal{K} \right),\end{aligned}\quad (3.26)$$

where $\mathcal{R}^{(3)}(\mathbf{x})$ is a pure third order integration constant. With the third-order curvature potential, the RHS of (3.1) at the third order in perturbations is then

$$\begin{aligned}\text{RHS}^{(3)}(t, \mathbf{x}) = & -\Delta\varphi^{(3)} + 3\nabla^i \varphi_2^{(2)} \nabla_i \mathcal{R} + 4\varphi_2^{(2)} \Delta \mathcal{R} + 4\mathcal{R} \Delta \varphi_2^{(2)} \\ & + 3\nabla^i \mathcal{R}^{(2)} \nabla_i \mathcal{R} + 4\mathcal{R}^{(2)} \Delta \mathcal{R} + 4\mathcal{R} \Delta \mathcal{R}^{(2)} - 3\mathcal{R} (3\nabla^i \mathcal{R} \nabla_i \mathcal{R} + 4\mathcal{R} \Delta \mathcal{R}) \\ & - a^2 H \left[-D_1 \Delta \nabla_i \mathcal{R} \nabla^i \left(\frac{\chi_1^{(2)}}{a^2} - \frac{\Delta^{-1}}{2} \dot{\mathcal{K}} \right) + \dot{D}_1 \nabla_i \mathcal{R} \nabla^i \delta^{(2)} \right] - 2a^2 H D_1 \dot{D}_1 \mathcal{R} \nabla^i \mathcal{R} \Delta \nabla_i \mathcal{R} \\ & - a^2 H \Delta \mathcal{R} \left[\dot{D}_1 \delta^{(2)} + D_1 \left(\dot{\delta}_1^{(2)} + \frac{\dot{\mathcal{K}}}{2} \right) \right] + \frac{a^2}{2} \dot{D}_1 \left[\frac{\Delta \mathcal{R}}{3} \left(\frac{\Delta \chi_1^{(2)}}{a^2} - \frac{\dot{\mathcal{K}}}{2} \right) - \nabla_i \nabla_j \mathcal{R} \nabla^i \nabla^j \left(\frac{\chi_1^{(2)}}{a^2} - \frac{\Delta^{-1}}{2} \dot{\mathcal{K}} \right) \right] \\ & + a^2 \dot{D}_1^2 \left[\mathcal{R} \nabla^i \nabla^j \mathcal{R} \nabla_i \nabla_j \mathcal{R} + \nabla^i \mathcal{R} \nabla^j \mathcal{R} \nabla_i \nabla_j \mathcal{R} - \frac{1}{3} \mathcal{R} (\Delta \mathcal{R})^2 - \frac{1}{3} \nabla^i \mathcal{R} \nabla_i \mathcal{R} \Delta \mathcal{R} \right] - \frac{\Delta \mathcal{R}}{3H\Sigma_1} \left(\dot{\delta}_1^{(2)} + \frac{\dot{\mathcal{K}}}{2} \right).\end{aligned}\quad (3.27)$$

From these we can first find the time-independent spatial function on the RHS which give rise to D_1 :

$$X_1^{(3)} = -\Delta \mathcal{R}^{(3)} - 2\Delta \left(\mathcal{R} \mathcal{R}^{(2)} \right) + 3\nabla^i \mathcal{R}^{(2)} \nabla_i \mathcal{R} + 4\mathcal{R}^{(2)} \Delta \mathcal{R} + 4\mathcal{R} \Delta \mathcal{R}^{(2)} - 3\mathcal{R} (3\nabla^i \mathcal{R} \nabla_i \mathcal{R} + 4\mathcal{R} \Delta \mathcal{R}), \quad (3.28)$$

which leads to the third-order extrinsic curvature perturbation proportional to D_1 :

$$\frac{\kappa_1^{(3)}(t, \mathbf{x})}{H f_1} = \delta_1^{(3)}(t, \mathbf{x}) = D_1(t) X_1^{(3)}(\mathbf{x}). \quad (3.29)$$

Those associated with the growth factors D_{2A} , D_{2B} and D_{2C} found in Section 3.2:

$$\begin{aligned}X_{2A}^{(3)} = & \frac{5}{7} \left\{ -\frac{\nabla^i \varphi_2^{(2)}}{D_1} \nabla_i \mathcal{R} + 2\frac{\varphi_2^{(2)}}{D_1} \Delta \mathcal{R} + 2\mathcal{R} \frac{\Delta \varphi_2^{(2)}}{D_1} + \nabla_i \left[-2\nabla^i \mathcal{R} \nabla^j \mathcal{R} \nabla_j \mathcal{R} - 2\mathcal{R} \nabla^i \nabla^j \mathcal{R} \nabla_j \mathcal{R} - 2\mathcal{R} \Delta \mathcal{R} \nabla^i \mathcal{R} \right. \right. \\ & \left. \left. + \frac{1}{2} \nabla^i \nabla^j \mathcal{R}^{(2)} \nabla_j \mathcal{R} + \frac{1}{2} \Delta \mathcal{R}_2 \nabla^i \mathcal{R} + \frac{1}{2} \nabla^i \nabla^j \mathcal{R} \nabla_j \left(H \Sigma_1 \chi_1^{(2)} \right) + \frac{1}{2} \Delta \mathcal{R} \nabla^i \left(H \Sigma_1 \chi_1^{(2)} \right) \right] \right\},\end{aligned}\quad (3.30)$$

$$\begin{aligned}X_{2B}^{(3)} = & \frac{2}{7} \left[\frac{\Delta \mathcal{R}}{6} \Delta \left(H \Sigma_1 \chi_1^{(2)} \right) - \frac{\nabla_i \nabla_j \mathcal{R}}{2} \nabla^i \nabla^j \left(H \Sigma_1 \chi_1^{(2)} \right) \right. \\ & \left. + \mathcal{R} \nabla^i \nabla^j \mathcal{R} \nabla_i \nabla_j \mathcal{R} + \nabla^i \mathcal{R} \nabla^j \mathcal{R} \nabla_i \nabla_j \mathcal{R} - \frac{1}{3} \mathcal{R} (\Delta \mathcal{R})^2 - \frac{1}{3} \nabla^i \mathcal{R} \nabla_i \mathcal{R} \Delta \mathcal{R} - \frac{\Delta \mathcal{R}}{3} X_1^{(2)} \right],\end{aligned}\quad (3.31)$$

$$X_{2C}^{(3)} = \frac{2}{7} \left[\Delta \nabla_i \mathcal{R} \nabla^i \left(H \Sigma_i \chi_1^{(2)} \right) - 2\mathcal{R} \nabla^i \mathcal{R} \Delta \nabla_i \mathcal{R} - \nabla_i \mathcal{R} \nabla^i X_1^{(2)} - 2\Delta \mathcal{R} X_1^{(2)} \right]. \quad (3.32)$$

Then we find the third-order solution with the second-order growth factors as

$$\delta_2^{(3)}(t, \mathbf{x}) = \sum_{I=A}^C D_{2I}(t) X_2 I^{(3)}(\mathbf{x}). \quad (3.33)$$

We also have new growth factors that become D_1^3 in the EdS universe:

$$D_{3D} \equiv \frac{9}{5} H \int dt D_1^3 f_1 \Sigma_1, \quad D_{3E} \equiv \frac{9}{2} H \int dt D_1^3 f_1, \quad D_{3F} \equiv \frac{9}{2} H \int dt D_1^3 f_1^2, \quad (3.34)$$

with the spatial functions associated with them:

$$X_{3D}^{(3)} = \frac{5}{36} \nabla_i \left(\frac{\nabla^i \nabla^j \varphi_2^{(2)}}{D_1} \nabla_j \mathcal{R} + \frac{\Delta \varphi_2^{(2)}}{D_1} \nabla^i \mathcal{R} \right) + \frac{5}{36} \nabla_i \{ (\nabla^i \nabla^j \mathcal{R} \Delta^{-1} \nabla_j + \Delta \mathcal{R} \Delta^{-1} \nabla^i) [\nabla^k \mathcal{R} \Delta \nabla_k \mathcal{R} + (\Delta \mathcal{R})^2] \} , \quad (3.35)$$

$$X_{3E}^{(3)} = \frac{1}{9} \{ \Delta^{-1} \nabla^i [\nabla^j \mathcal{R} \Delta \nabla_j \mathcal{R} + (\Delta \mathcal{R})^2] \Delta \nabla_i \mathcal{R} + [\nabla^j \mathcal{R} \Delta \nabla_j \mathcal{R} + (\Delta \mathcal{R})^2] \Delta \mathcal{R} \} , \quad (3.36)$$

$$X_{3F}^{(3)} = -\frac{1}{18} (\nabla^i \nabla^j \mathcal{R} \Delta^{-1} \nabla_i \nabla_j - \Delta \mathcal{R}) [\nabla^k \mathcal{R} \Delta \nabla_k \mathcal{R} + (\Delta \mathcal{R})^2] . \quad (3.37)$$

The associated third-order solution, which constitutes one part of $\delta_3^{(3)}$ is

$$\delta_3^{(3)}(t, \mathbf{x}) \supset \sum_{I=D}^F D_{3I}(t) X_{3I}^{(3)}(\mathbf{x}) . \quad (3.38)$$

Finally, the growth factors coming from $\delta_{2I}^{(2)}$ ($I = A, B, C$) also scale as D_1^3 in the EdS universe:

$$D_{3Ia} \equiv \frac{9}{5} H \int dt D_1 f_1 \Sigma_1 D_{2I} , \quad D_{3Ib} \equiv \frac{9}{4} H \int dt D_1 D_{2I} f_{2I} , \quad (3.39)$$

$$D_{3Ic} \equiv \frac{9}{2} H \int dt D_1 f_1 D_{2I} , \quad D_{3Id} \equiv \frac{9}{4} H \int dt D_1 f_1 D_{2I} f_{2I} , \quad (3.40)$$

with the corresponding spatial functions:

$$X_{3Ia}^{(3)} = -\frac{5}{18} \nabla_i \left[(\nabla^i \nabla^j \mathcal{R} \Delta^{-1} \nabla_j + \Delta \mathcal{R} \Delta^{-1} \nabla^i) X_{2I}^{(2)} \right] , \quad (3.41)$$

$$X_{3Ib}^{(3)} = -\frac{4}{9} \nabla_i \left(\Delta \mathcal{R} \Delta^{-1} \nabla^i X_{2I}^{(2)} \right) , \quad (3.42)$$

$$X_{3Ic}^{(3)} = -\frac{2}{9} \nabla_i \left(X_{2I}^{(2)} \nabla^i \mathcal{R} \right) , \quad (3.43)$$

$$X_{3Id}^{(3)} = \frac{2}{9} (\nabla^i \nabla^j \mathcal{R} \Delta^{-1} \nabla_i \nabla_j - \Delta \mathcal{R}) X_{2I}^{(2)} . \quad (3.44)$$

These give the other part of $\delta_3^{(3)}$:

$$\delta_3^{(3)}(t, \mathbf{x}) \supset \sum_{I=A}^C \sum_{i=a}^d D_{3Ii}(t) X_{3Ii}^{(3)}(\mathbf{x}) . \quad (3.45)$$

The full third-order solution is the sum of (3.29), (3.33), (3.38) and (3.45):

$$\delta^{(3)}(t, \mathbf{x}) = \delta_1^{(3)} + \delta_2^{(3)} + \delta_3^{(3)} = D_1 X_1^{(3)} + \sum_{I=A}^C D_{2I} X_{2I}^{(3)} + \sum_{I=D}^F D_{3I} X_{3I}^{(3)} + \sum_{I=A}^C \sum_{i=a}^d D_{3Ii} X_{3Ii}^{(3)} . \quad (3.46)$$

This analytic third-order solution is one of the main results of this article.

3.4 Full third-order solutions in the EdS universe

In the EdS universe, it is only the matter density that drives the Hubble expansion and the growth of perturbations, thus providing the simplest example and consistency checks, to which we can compare our analytic solutions in a Λ CDM universe.

With $\Omega_m = 1$, the Hubble parameter is $H = 2/(3t)$, and all the quantities are scale-free in their time-dependence. The RHS of (3.1) has the simple time-dependence:

$$\text{RHS}^{(n)}(t, \mathbf{x}) \propto \frac{1}{\mathcal{H}^{2(n-1)}} ; \quad T_1(t) = 1 , \quad T_2(t) = \frac{14}{25\mathcal{H}^2} , \quad T_3(t) = \frac{36}{75\mathcal{H}^4} , \quad (3.47)$$

regardless of its order in perturbations. Therefore, all the growth factors associated with each $\text{RHS}^{(n)}$ are all identical, and they can be analytically integrated as

$$D_1(t) = H \int dt \frac{1}{\mathcal{H}^2} = \frac{2}{5\mathcal{H}^2}, \quad f_1 = 1, \quad \Sigma_1 = \frac{5}{2}, \quad (3.48)$$

$$D_2(t) = H \int dt \frac{14}{25\mathcal{H}^4} = \frac{2^2}{5^2\mathcal{H}^4} = D_1^2, \quad f_2 = 2, \quad \Sigma_2 = \frac{7}{4}, \quad (3.49)$$

$$D_3(t) = H \int dt \frac{36}{75\mathcal{H}^6} = \frac{2^3}{5^3\mathcal{H}^6} = D_1^3, \quad f_3 = 3, \quad \Sigma_3 = \frac{3}{2}. \quad (3.50)$$

The Newtonian solutions in the EdS universe are then

$$\delta_1^{(1)}(t, \mathbf{x}) = \frac{\kappa_1^{(1)}(t, \mathbf{x})}{H} = -D_1(t)\Delta\mathcal{R}(\mathbf{x}), \quad (3.51)$$

$$\delta_2^{(2)}(t, \mathbf{x}) = \frac{D_1^2(t)}{7} \left[5(\Delta\mathcal{R})^2 + 2\nabla^i\nabla^j\mathcal{R}\nabla_i\nabla_j\mathcal{R} + 7\nabla_i\mathcal{R}\Delta\nabla^i\mathcal{R} \right], \quad (3.52)$$

$$\frac{\kappa_2^{(2)}(t, \mathbf{x})}{H} = \frac{D_1^2(t)}{7} \left[3(\Delta\mathcal{R})^2 + 4\nabla^i\nabla^j\mathcal{R}\nabla_i\nabla_j\mathcal{R} + 7\nabla_i\mathcal{R}\Delta\nabla^i\mathcal{R} \right], \quad (3.53)$$

$$\delta_3^{(3)}(t, \mathbf{x}) = -\frac{D_1(t)}{18} \left[2\Delta \left(\nabla_i\mathcal{R}\Delta^{-1}\nabla^i\frac{\kappa_2^{(2)}}{H} \right) + 7\nabla^i \left(\Delta^{-1}\nabla_i\frac{\kappa_2^{(2)}}{H}\Delta\mathcal{R} \right) + 7\nabla^i \left(\delta_2^{(2)}\nabla_i\mathcal{R} \right) \right], \quad (3.54)$$

$$\frac{\kappa_3^{(3)}(t, \mathbf{x})}{H} = -\frac{D_1(t)}{6} \left[2\Delta \left(\nabla_i\mathcal{R}\Delta^{-1}\nabla^i\frac{\kappa_2^{(2)}}{H} \right) + \nabla^i \left(\Delta^{-1}\nabla_i\frac{\kappa_2^{(2)}}{H}\Delta\mathcal{R} \right) + \nabla^i \left(\delta_2^{(2)}\nabla_i\mathcal{R} \right) \right], \quad (3.55)$$

and the relativistic solutions are

$$\delta_1^{(2,3)}(t, \mathbf{x}) = \frac{\kappa_1^{(2,3)}(t, \mathbf{x})}{H} = D_1(t) \left[\frac{3}{2}\nabla^i\mathcal{R}\nabla_i\mathcal{R} + 4\mathcal{R}\Delta\mathcal{R} - 3\mathcal{R}(3\nabla^i\mathcal{R}\nabla_i\mathcal{R} + 4\mathcal{R}\Delta\mathcal{R}) \right], \quad (3.56)$$

$$\begin{aligned} \delta_2^{(3)}(t, \mathbf{x}) = & \frac{D_1^2(t)}{7} \left[\frac{8}{3}\mathcal{R}(\Delta\mathcal{R})^2 - 8\mathcal{R}\nabla^i\nabla^j\mathcal{R}\nabla_i\nabla_j\mathcal{R} - 14\mathcal{R}\nabla^i\mathcal{R}\Delta\nabla_i\mathcal{R} - 8\nabla^i\nabla^j\mathcal{R}\nabla_i\mathcal{R}\nabla_j\mathcal{R} \right. \\ & + \frac{8}{3}\nabla^i\mathcal{R}\nabla_i\mathcal{R}\Delta\mathcal{R} + \left(7\Delta\nabla_i\mathcal{R}\nabla^i + 4\nabla_i\nabla_j\mathcal{R}\nabla^i\nabla^j - \frac{4}{3}\Delta\mathcal{R}\Delta \right) \left(D_1^{-1}\Delta^{-1}\delta_1^{(2)} + \frac{5}{2}H\Delta\chi_1^{(2)} \right) \\ & \left. - (10\Delta\mathcal{R} + 7\Delta\nabla_i\mathcal{R}\Delta^{-1}\nabla^i + 7\nabla_i\mathcal{R}\nabla^i + 4\nabla_i\nabla_j\mathcal{R}\Delta^{-1}\nabla^i\nabla^j) D_1^{-1}\delta_1^{(2)} \right], \quad (3.57) \end{aligned}$$

$$\begin{aligned} \frac{\kappa_2^{(3)}(t, \mathbf{x})}{H} = & \frac{D_1^2(t)}{7} \left[\frac{16}{3}\mathcal{R}(\Delta\mathcal{R})^2 - 16\nabla^i\nabla^j\mathcal{R}\nabla_i\nabla_j\mathcal{R} - 14\nabla^i\mathcal{R}\Delta\nabla_i\mathcal{R} - 16\nabla^i\nabla^j\mathcal{R}\nabla_i\mathcal{R}\nabla_j\mathcal{R} \right. \\ & + \frac{16}{3}\nabla^i\mathcal{R}\nabla_i\mathcal{R}\Delta\mathcal{R} + \left(7\Delta\nabla_i\mathcal{R}\nabla^i + 8\nabla_i\nabla_j\mathcal{R}\nabla^i\nabla^j - \frac{8}{3}\Delta\mathcal{R}\Delta \right) \left(D_1^{-1}\Delta^{-1}\delta_1^{(2)} + \frac{5}{2}H\Delta\chi_1^{(2)} \right) \\ & \left. - (6\Delta\mathcal{R} + 7\Delta\nabla_i\mathcal{R}\Delta^{-1}\nabla^i + 7\nabla_i\mathcal{R}\nabla^i + 8\nabla_i\nabla_j\mathcal{R}\Delta^{-1}\nabla^i\nabla^j) D_1^{-1}\delta_1^{(2)} \right]. \quad (3.58) \end{aligned}$$

In presenting the above solutions, we assumed the initial condition $\mathcal{R}(\mathbf{x})$ is at the linear order in perturbations. But in principle, the initial condition can be treated as a nonlinear perturbation variable, such that $\delta_1 = -D_1\Delta\mathcal{R}$ also contributes to n -th order in perturbation, for instance, if $\mathcal{R} = \mathcal{R}^{(n)}$ as is explicit in (3.19) and (3.28).

4 Comparison to previous works in general relativity

In this section, we compare our analytic solutions in Section 3 to the solution derived in [17, 18, 19]. The full relativistic matter density fluctuation and its one-loop power spectrum [17, 18] as well as one-loop bispectrum [19] were computed under the same gauge condition, but by assuming the EdS universe. However, the solution

in [17, 18, 19] differs from ours in the relativistic corrections. The critical differences in the previous works are that (a) the initial condition is set by the density fluctuation rather than the curvature perturbation φ , and that (b) it was assumed to be at the linear order in perturbation, i.e., $\delta(\mathbf{x}, t_i) = \delta_1^{(1)}(\mathbf{x}, t_i)$ at some early time t_i . Since the curvature perturbation spectrum is set up by inflation in the early Universe and is conserved on super-horizon scales, it is more natural to set up the initial condition for the nonlinear evolution of the matter density fluctuation with the curvature perturbation as in the current study. However, their set-up with δ is just fine, as we show below that the ADM energy constraint equation (4.15) leads to the equivalent initial condition set up by the curvature perturbation. The real difference lies in (b), in clear disagreement with our finding in Section 3. Here we derive the previous work in configuration space, rather than in Fourier space as was done in [17, 18, 19], and show how these differences play a role in connecting two solutions.

4.1 Derivation of previous works in configuration space

While the relativistic dynamical equations are identical, the approach to the solution in previous works focuses on the main dynamical variables δ and κ , rather than \mathcal{R} , which play a role of supplementing the Newtonian dynamical equations with relativistic corrections. The conservation equation (2.9) and the Raychaudhuri equation (2.10) are explicitly expanded up to the third order in perturbations by using (2.13) and (2.15) as

$$\dot{\delta} - \kappa = -\frac{1}{a^2}(1 - 2\varphi)\nabla^i\chi\nabla_i\delta + \delta\kappa, \quad (4.1)$$

$$\begin{aligned} \dot{\kappa} + 2H\kappa - 4\pi G\bar{\rho}_m\delta = & -\frac{1}{a^2}(1 - 2\varphi)\nabla^i\chi\nabla_i\kappa + \frac{1}{3}\kappa^2 + \frac{1}{a^4}\left[\nabla_i\nabla_j\chi\nabla^i\nabla^j\chi - \frac{1}{3}(\Delta\chi)^2\right](1 - 4\varphi) \\ & - \frac{4}{a^4}\left(\nabla^i\nabla^j\chi - \frac{1}{3}\bar{g}^{ij}\Delta\chi\right)\nabla_i\chi\nabla_j\varphi. \end{aligned} \quad (4.2)$$

These dynamical equations are solved in conjunction with the ADM energy constraint equation at the linear order

$$\frac{3}{2}H^2\delta + H\kappa + \frac{\Delta}{a^2}\varphi = 0, \quad (4.3)$$

and the second-order ADM momentum constraint equation (2.16). We will only use the linear-order curvature perturbation φ and the second-order scalar shear χ in this section, complementing the dynamical equations for δ and κ .

With the knowledge of the time dependence of the EdS solutions in Section 3.4, we seek solutions of the dynamical equations (4.1) and (4.2) by explicitly removing their time-dependence. Up to third order, the density fluctuation δ and the perturbation to the extrinsic curvature κ are parametrized as

$$\delta(t, \mathbf{x}) \equiv \frac{c_1(\mathbf{x})}{\mathcal{H}^2} + \frac{c_2(\mathbf{x})}{\mathcal{H}^4} + \frac{c_3(\mathbf{x})}{\mathcal{H}^6}, \quad \frac{\kappa(t, \mathbf{x})}{H} \equiv \frac{d_1(\mathbf{x})}{\mathcal{H}^2} + \frac{d_2(\mathbf{x})}{\mathcal{H}^4} + \frac{d_3(\mathbf{x})}{\mathcal{H}^6}, \quad (4.4)$$

where $c_i(\mathbf{x})$ and $d_i(\mathbf{x})$ are time-independent spatial functions and they vanish when the n -th order perturbation is considered, if $i > n$. Note that we have grouped δ and κ according to the same time dependence, *not* to the perturbation order, as denoted by subscripts. For example, $c_1(\mathbf{x})$ may contain non-linear perturbation terms as we will show right below. Inspecting the ADM momentum constraint equation (2.16), we can also parametrized the scalar shear up to the second order in perturbations as

$$H\chi \equiv e_1(\mathbf{x}) + \frac{e_2(\mathbf{x})}{\mathcal{H}^2}, \quad (4.5)$$

and of course the curvature perturbation $\varphi(\mathbf{x})$ is time-independent at the linear order. Comparing to the notation in Section 3, we can readily identify the correspondence:

$$\delta_i(t, \mathbf{x}) = \frac{c_i(\mathbf{x})}{\mathcal{H}^{2i}}, \quad \chi_1 = \frac{e_1(\mathbf{x})}{H}, \quad \chi_2 = \frac{e_2(\mathbf{x})}{a^2 H^3}. \quad (4.6)$$

Armed with the parametrized solutions, the conservation equation (4.1) yields a set of *algebraic* equations

$$\begin{aligned} c_1 - d_1 &= 0, \\ 2c_2 - d_2 &= -(1 - 2\varphi)\nabla^i e_1 \nabla_i c_1 + c_1 d_1, \\ 3c_3 - d_3 &= -(1 - 2\varphi)(\nabla^i e_1 \nabla_i c_2 + \nabla^i e_2 \nabla_i c_1) + c_1 d_2 + c_2 d_1, \end{aligned} \quad (4.7)$$

and we now simply set $c_1 = d_1 \equiv c$ at all orders in perturbation. Similarly, the Raychaudhuri equation (4.2) provides

$$\begin{aligned} \frac{5d_2 - 3c_2}{2} &= -(1 - 2\varphi)\nabla^i e_1 \nabla_i d_1 + \frac{1}{3}d_1^2 + (1 - 4\varphi) \left[\nabla_i \nabla_j e_1 \nabla^i \nabla^j e_1 - \frac{1}{3}(\Delta e_1)^2 \right] \\ &\quad - 4 \left(\nabla^i \nabla^j e_1 - \frac{1}{3}\bar{g}^{ij} \Delta e_1 \right) \nabla_i e_1 \nabla_j \varphi , \\ \frac{7d_3 - 3c_3}{2} &= -\nabla^i e_1 \nabla_i d_2 - \nabla^i e_2 \nabla_i d_1 + \frac{2}{3}d_1 d_2 + 2\nabla^i \nabla^j e_1 \nabla_i \nabla_j e_2 - \frac{2}{3}\Delta e_1 \Delta e_2 , \end{aligned} \quad (4.8)$$

and the ADM momentum constraint provides the supplementary equation for the scalar shear³

$$\begin{aligned} d_1 + \Delta e_1 &= 2\varphi \Delta e_1 - \nabla^i \varphi \nabla_i e_1 + \frac{3}{2}\Delta^{-1} \nabla^i (\nabla_i e_1 \Delta \varphi + \nabla_j \nabla_i \varphi \nabla^j e_1) \equiv \Delta \Psi , \\ d_2 + \Delta e_2 &= 0 . \end{aligned} \quad (4.9)$$

Therefore, the solutions to the algebraic equations are

$$\begin{aligned} c_2 &= \frac{1}{7}c^2 \left(5 + \frac{8}{3}\varphi \right) + (1 - 2\varphi)\nabla^i \Delta^{-1} c \nabla_i c + \frac{2}{7}(1 - 4\varphi)\nabla_i \nabla_j \Delta^{-1} c \nabla^i \nabla^j \Delta^{-1} c \\ &\quad - \frac{8}{7} \left(\nabla^i \nabla^j \Delta^{-1} c - \frac{1}{3}\bar{g}^{ij} c \right) \nabla_i \Delta^{-1} c \nabla_j \varphi - \left(\nabla_i c \nabla^i + \frac{4}{7}\nabla_i \nabla_j \Delta^{-1} c \nabla^i \nabla^j - \frac{4}{21}c \Delta \right) \Psi , \end{aligned} \quad (4.10)$$

$$c_3 = \frac{1}{18} \left[7\nabla^i (c_2 \Delta^{-1} \nabla_i c) + 2\Delta (\Delta^{-1} \nabla_i c \nabla^i \Delta^{-1} d_2) + 7\nabla^i (c \nabla_i \Delta^{-1} d_2) \right] , \quad (4.11)$$

$$\begin{aligned} d_2 &= \frac{1}{7}c^2 \left(3 + \frac{16}{3}\varphi \right) + (1 - 2\varphi)\nabla^i \Delta^{-1} c \nabla_i c + \frac{4}{7}(1 - 4\varphi)\nabla_i \nabla_j \Delta^{-1} c \nabla^i \nabla^j \Delta^{-1} c \\ &\quad - \frac{16}{7} \left(\nabla^i \nabla^j \Delta^{-1} c - \frac{1}{3}\bar{g}^{ij} c \right) \nabla_i \Delta^{-1} c \nabla_j \varphi - \left(\nabla_i c \nabla^i + \frac{8}{7}\nabla_i \nabla_j \Delta^{-1} c \nabla^i \nabla^j - \frac{8}{21}c \Delta \right) \Psi , \end{aligned} \quad (4.12)$$

$$d_3 = \frac{1}{6} \left[\nabla^i (c_2 \Delta^{-1} \nabla_i c) + 2\Delta (\Delta^{-1} \nabla_i c \nabla^i \Delta^{-1} d_2) + \nabla^i (c \nabla_i \Delta^{-1} d_2) \right] . \quad (4.13)$$

This completes our derivation of previous work in configuration space. It is noted that the solution was derived [17, 18, 19] in Fourier space (see Section 5.3), and the initial condition set up with the coefficient c of the density fluctuation is assumed to be at the linear-order in perturbations.

4.2 Comparison to our solution

Compared to our analytic solutions in (3.51)–(3.58), these solutions are expressed in terms of the coefficient $c(\mathbf{x})$ of the density fluctuation in proportion to $1/\mathcal{H}^2$, rather than the curvature perturbation \mathcal{R} . Given the ADM energy constraint equation (4.3) at the *linear order* in perturbation, the relation of this coefficient to the curvature perturbation is

$$c = c_1 = d_1 = -\frac{2}{5}\Delta\varphi = -\frac{2}{5}\Delta\mathcal{R} . \quad (4.14)$$

With this relation, we can easily recover our analytic solutions $\delta_i^{(i)}$ and $\kappa_i^{(i)}$ in (3.51)–(3.55), corresponding to the standard Newtonian solutions, while there remain the differences in the relativistic corrections $\delta_i^{(j)}$ and $\kappa_i^{(j)}$ for $j > i$.

The reason for this difference is that the density fluctuation δ_1 (or the coefficient c) is treated as the linear-order perturbation $\delta^{(1)}$ — in our derivation we made *no assumption* about the perturbation orders of all the coefficients, while only the linear-order ADM energy constraint is used to convert the parametrized solutions

³ Here, Ψ is identical to the following second-order quantity:

$$\Psi^{(2)}(\mathbf{x}) \equiv \frac{\delta_1^{(2)}(t, \mathbf{x})}{D_1(t)} + H\Sigma_1 \Delta\chi_1^{(2)}(t, \mathbf{x}) .$$

and compare to our analytic solutions in Section 3. Using the full ADM energy constraint equation (2.11) and taking the limit $t \rightarrow 0$, we derive the nonlinear relation

$$c(\mathbf{x}) = \frac{2}{5} \left[-\Delta \mathcal{R} + \frac{3}{2} \nabla^i \mathcal{R} \nabla_i \mathcal{R} + 4\mathcal{R} \Delta \mathcal{R} - 3\mathcal{R} (3\nabla^i \mathcal{R} \nabla_i \mathcal{R} + 4\mathcal{R} \Delta \mathcal{R}) \right], \quad (4.15)$$

and the solutions up to third order in perturbation in previous work are readily obtained as

$$\delta_1 = \frac{\kappa_1}{H} = \frac{c}{\mathcal{H}^2} = \frac{2}{5\mathcal{H}^2} \left[-\Delta \mathcal{R} + \frac{3}{2} \nabla^i \mathcal{R} \nabla_i \mathcal{R} + 4\mathcal{R} \Delta \mathcal{R} - 3\mathcal{R} (3\nabla^i \mathcal{R} \nabla_i \mathcal{R} + 4\mathcal{R} \Delta \mathcal{R}) \right], \quad (4.16)$$

identical to our analytic solution in Section 3. Using the relation to the second order in perturbations, we can repeat this exercise to recover the relativistic corrections in $\delta_2^{(3)}$ and $\kappa_2^{(3)}$. This proves the equivalence of the solutions in (4.10)–(4.13) to the analytic solutions in Section 3. However, the density fluctuation δ_1 (or the coefficient c) is *not* a linear-order perturbation, as is apparent in (4.15). Therefore, the Fourier kernels derived in [17, 18, 19] are valid *only under the assumption* that the initial condition set up by the density fluctuation δ_1 is linear order in perturbations. We will provide the complete Fourier kernels in Section 5.3.

5 Comparison to the standard Newtonian perturbation theory

In this section, we compare our relativistic solutions to the standard Newtonian solutions. This provides insights to understand the connection of the relativistic dynamics to the Newtonian dynamics. We also derive the Fourier kernels for the relativistic solutions.

5.1 Dynamical equations of motion

As noted in the previous sections, the relativistic solutions in our gauge condition closely resemble the standard Newtonian ones. In the comoving gauge, the dynamical equations of motion are shown to be identical to the Newtonian ones to the second order in perturbations [31, 32] with the relativistic effects appearing only from the third order [33]. This gauge condition is later shown [13] to correspond to the proper-time hypersurface of nonrelativistic matter flows. In the proper-time hypersurface, the local observer moving with the nonrelativistic matter flows can measure the energy density in its rest frame, providing the most natural description of the matter density fluctuation.

Following this approach, we compare the relativistic dynamical equations (4.1) and (4.2) with those in the Newtonian dynamics by identifying proper correspondences between the relativistic and Newtonian dynamics. In the standard Newtonian perturbation theory, the velocity \mathbf{v}_N of the flow is often expressed in terms of the velocity divergence field θ_N :

$$\theta_N \equiv \frac{1}{a} \nabla \cdot \mathbf{v}_N, \quad (5.1)$$

where the subscript N is used to indicate the quantity is a Newtonian variable. Since in general relativistic approach with our gauge condition, κ is the perturbation in the expansion of the local observer, we define the (nonlinear) relativistic velocity \mathbf{v} of the observer as

$$-\kappa \equiv \frac{1}{a} \nabla \cdot \mathbf{v}. \quad (5.2)$$

This “velocity” is defined only in relation to κ , which is *not* identical to the spatial component of the four velocity in (2.2).

With this identification of velocity, the relativistic dynamical equations (4.1) and (4.2) can be rewritten in

terms of the matter density fluctuation and the velocity as

$$\dot{\delta} + \frac{1}{a} \nabla \cdot \mathbf{v} = -\frac{1}{a} \nabla \cdot (\mathbf{v} \delta) + \frac{2\varphi}{a} \nabla \delta \cdot \mathbf{v} - \frac{1}{a} \nabla \delta \cdot \nabla \Delta^{-1} \left[2\varphi \nabla \cdot \mathbf{v} - (\mathbf{v} \cdot \nabla) \varphi + \frac{3}{2} \Delta^{-1} \nabla \cdot ((\mathbf{v} \cdot \nabla) \nabla \varphi + \mathbf{v} \Delta \varphi) \right], \quad (5.3)$$

$$\nabla \cdot \dot{\mathbf{v}} + H \nabla \cdot \mathbf{v} + \frac{3H^2}{2} a \Omega_m \delta = -\frac{1}{a} \nabla \cdot [(\mathbf{v} \cdot \nabla) \mathbf{v}] - \frac{2}{3a} \varphi (\mathbf{v} \cdot \nabla) (\nabla \cdot \mathbf{v}) + \frac{4}{a} \nabla \cdot \left[\varphi \left((\mathbf{v} \cdot \nabla) \mathbf{v} - \frac{1}{3} \mathbf{v} (\nabla \cdot \mathbf{v}) \right) \right] + \frac{1}{a} \left[\mathbf{v} \cdot \nabla + \frac{2}{3} \nabla \cdot \mathbf{v} - \Delta \left((\mathbf{v} \cdot \nabla) \Delta^{-1} \right) \right] \left[2\varphi \nabla \cdot \mathbf{v} - (\mathbf{v} \cdot \nabla) \varphi + \frac{3}{2} \Delta^{-1} \nabla \cdot ((\mathbf{v} \cdot \nabla) \nabla \varphi + \mathbf{v} \Delta \varphi) \right]. \quad (5.4)$$

It is now evident that the relativistic dynamics in our gauge condition with the proper correspondence between (δ, \mathbf{v}) and (δ_N, \mathbf{v}_N) follows the standard Newtonian equations of motion up to the second order in perturbations and the relativistic corrections that contain the curvature perturbation φ appear only at the third order in the equations of motion. From below, we refer to the “Newtonian dynamical equations” as (5.3) and (5.4) without φ terms, with the identification $\delta \rightarrow \delta_N$ and $\mathbf{v} \rightarrow \mathbf{v}_N$. Note, however, that in the Newtonian dynamics, there is no constraint beyond the equation of motion such that the initial condition $c(\mathbf{x})$ in Section 4.1 is rather unconstrained, as opposed to the case in the relativistic dynamics due to the full ADM energy constraint equation (4.15). Finally, the linear-order ADM energy constraint equation (4.3) can be written as, using the ADM momentum constraint (2.16) to linear order to replace κ with χ ,

$$\frac{3}{2} H^2 \Omega_m \delta = -\frac{\Delta}{a^2} (\varphi - H\chi) \equiv -\frac{\Delta}{a^2} \varphi_\chi, \quad (5.5)$$

indicating that we may identify the Newtonian potential $\Phi_N = -\varphi_\chi$, where φ_χ is the linear-order curvature potential in the zero shear gauge.

5.2 Standard perturbation theory

Here we briefly summarize the key equations for deriving the standard Fourier kernels and their recurrence relations. A comprehensive review on this topic can be found in [30] (and references therein).

By assuming the separability of the time and the spatial dependences, the standard perturbation theory (SPT) takes a perturbative approach to the nonlinear solution:

$$\delta_N(t, \mathbf{k}) \equiv \sum_{n=1}^{\infty} D^n(t) \left[\prod_i^n \int \frac{d^3 \mathbf{q}_i}{(2\pi)^3} \hat{\delta}(\mathbf{q}_i) \right] (2\pi)^3 \delta^D(\mathbf{k} - \mathbf{q}_{12\dots n}) F_n^{(s)}(\mathbf{q}_1, \dots, \mathbf{q}_n) \equiv \sum_{n=1}^{\infty} D^n(t) \delta^{(n)}(\mathbf{k}), \quad (5.6)$$

$$\frac{\theta_N(t, \mathbf{k})}{H f_1} \equiv \sum_{n=1}^{\infty} D^n(t) \left[\prod_i^n \int \frac{d^3 \mathbf{q}_i}{(2\pi)^3} \hat{\delta}(\mathbf{q}_i) \right] (2\pi)^3 \delta^D(\mathbf{k} - \mathbf{q}_{12\dots n}) G_n^{(s)}(\mathbf{q}_1, \dots, \mathbf{q}_n) \equiv \sum_{n=1}^{\infty} D^n \theta^{(n)}(\mathbf{k}), \quad (5.7)$$

where δ^D is the Dirac delta function, $\mathbf{q}_{12\dots n} \equiv \mathbf{q}_1 + \dots + \mathbf{q}_n$, $\delta^{(n)}(\mathbf{k})$ and $\theta^{(n)}(\mathbf{k})$ are time-independent n -th order perturbations, $F_n^{(s)}$ and $G_n^{(s)}$ are the SPT kernels symmetrized over its arguments. The (dimensionless) Newtonian linear-order growth factor $D(t) \equiv D_1(t)/D_1(t_0)$ is normalized to unity at some early epoch t_0 when the nonlinearities are ignored and satisfies the differential equation $\ddot{D} + 2H\dot{D} - 4\pi G \bar{\rho}_m D = 0$. The initial linear density perturbation is set up in terms of which the perturbative expansion is given, $\delta_N(t_0, \mathbf{k}) \equiv \delta_1^{(1)}(t_0, \mathbf{k}) \equiv \hat{\delta}(\mathbf{k})$. With these decompositions in the Fourier space, the LHS of the Newtonian dynamical equations become

$$\dot{\delta}_N + \theta_N = H f_1 \sum_{n=1}^{\infty} D^n \left(n \delta^{(n)} - \theta^{(n)} \right), \quad (5.8)$$

$$\dot{\theta}_N + 2H\theta_N - 4\pi G \bar{\rho}_m \delta_N = H^2 f_1^2 \sum \frac{D^n}{2} \left[(1 + 2n) \theta^{(n)} - 3 \delta^{(n)} \right],$$

where we adopted the usual assumption $\Omega_m = f_1 = 1$ in SPT and utilized the relation between the growth factor and the growth rate $\dot{D} = H D f_1$. The RHS of the Newtonian dynamical equations are the convolution in

the Fourier space:

$$\begin{aligned} \left[-\frac{1}{a} \nabla \cdot (\delta_N \mathbf{v}_N) \right] (\mathbf{k}) &= \int \frac{d^3 \mathbf{Q}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{Q}_2}{(2\pi)^3} (2\pi)^3 \delta^D(\mathbf{k} - \mathbf{Q}_{12}) \alpha_{12} \theta_N(\mathbf{Q}_1, t) \delta_N(\mathbf{Q}_2, t) \equiv H f_1 \sum_{n=1}^{\infty} D^n A_n(\mathbf{k}) , \\ \left\{ \frac{1}{a^2} \nabla \cdot [(\mathbf{v}_N \cdot \nabla) \mathbf{v}_N] \right\} (\mathbf{k}) &= \int \frac{d^3 \mathbf{Q}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{Q}_2}{(2\pi)^3} (2\pi)^3 \delta^D(\mathbf{k} - \mathbf{Q}_{12}) \beta_{12} \theta_N(\mathbf{Q}_1, t) \theta_N(\mathbf{Q}_2, t) \equiv H^2 f_1^2 \sum_{n=1}^{\infty} D^n B_n(\mathbf{k}) , \end{aligned} \quad (5.9)$$

where the vertex functions are defined as

$$\alpha_{12} \equiv \alpha(\mathbf{Q}_1, \mathbf{Q}_2) \equiv 1 + \frac{\mathbf{Q}_1 \cdot \mathbf{Q}_2}{Q_1^2} \quad \text{and} \quad \beta_{12} \equiv \beta(\mathbf{Q}_1, \mathbf{Q}_2) \equiv \frac{|\mathbf{Q}_1 + \mathbf{Q}_2|^2 \mathbf{Q}_1 \cdot \mathbf{Q}_2}{2Q_1^2 Q_2^2} , \quad (5.10)$$

and the n -th order perturbation kernels $A_n(\mathbf{k})$ and $B_n(\mathbf{k})$ are

$$\begin{aligned} A_n(\mathbf{k}) &= \left[\prod_i^n \int \frac{d^3 \mathbf{q}_i}{(2\pi)^3} \hat{\delta}(\mathbf{q}_i) \right] (2\pi)^3 \delta^D(\mathbf{k} - \mathbf{q}_{12\dots n}) \sum_{i=1}^{n-1} \alpha_{12} G_i(\mathbf{q}_1, \dots, \mathbf{q}_i) F_{n-i}(\mathbf{q}_{i+1}, \dots, \mathbf{q}_n) , \\ B_n(\mathbf{k}) &= \left[\prod_i^n \int \frac{d^3 \mathbf{q}_i}{(2\pi)^3} \hat{\delta}(\mathbf{q}_i) \right] (2\pi)^3 \delta^D(\mathbf{k} - \mathbf{q}_{12\dots n}) \sum_{i=1}^{n-1} \beta_{12} G_i(\mathbf{q}_1, \dots, \mathbf{q}_i) G_{n-i}(\mathbf{q}_{i+1}, \dots, \mathbf{q}_n) , \end{aligned} \quad (5.11)$$

with $\mathbf{Q}_1 = \mathbf{q}_{1\dots i}$ and $\mathbf{Q}_1 + \mathbf{Q}_2 = \mathbf{k}$.

Therefore, the two Newtonian dynamical equations become algebraic equations without time-dependence:

$$n\delta^{(n)} - \theta^{(n)} = A_n , \quad (1 + 2n)\theta^{(n)} - 3\delta^{(n)} = 2B_n , \quad (5.12)$$

and the well-known recurrence formulas for the solutions are

$$\delta^{(n)} = \frac{(1 + 2n)A_n + 2B_n}{(2n + 3)(n - 1)} \quad \text{and} \quad \theta^{(n)} = \frac{3A_n + 2nB_n}{(2n + 3)(n - 1)} , \quad (5.13)$$

and similarly so for the SPT kernels

$$\begin{aligned} F_n &= \sum_{i=1}^{n-1} \frac{G_i}{(2n + 3)(n - 1)} [(1 + 2n)\alpha_{12} F_{n-i} + 2\beta_{12} G_{n-i}] , \\ G_n &= \sum_{i=1}^{n-1} \frac{G_i}{(2n + 3)(n - 1)} [3\alpha_{12} F_{n-i} + 2n\beta_{12} G_{n-i}] , \end{aligned} \quad (5.14)$$

with $F_1 = G_1 = 1$. Using the recurrence relations (5.14), the SPT kernels $F_n \sim G_n \propto k^2$ for $n > 1$ in the limit $k \rightarrow 0$, with the individual momentum \mathbf{q}_i held finite. This originates from the momentum conservation of the nonlinear evolution.

5.3 Relativistic effects in the density and velocity fluctuations

As emphasized, the relativistic dynamical equations (4.1) and (4.2) are identical to the standard Newtonian equations up to the second order terms [see also (5.3) and (5.4)], and the relativistic terms ($\sim \varphi$) appear only in the third order terms in the RHS of the dynamical equations. the Fourier decomposition of δ_N and θ_N in Section 5.2 is valid for δ and κ , but the relativistic corrections need to be further supplemented to the standard Newtonian solutions. Since the curvature perturbation φ is time-independent at the linear order, the time-dependences of these relativistic corrections in the RHS of the dynamical equations are

$$\frac{1}{a^2} \chi \delta \varphi \sim \dot{D} D \hat{\delta}^2 \varphi \sim H f_1 D^2 \hat{\delta}^3 , \quad \frac{1}{a^2} \chi \kappa \varphi \sim \dot{D}^2 \hat{\delta}^2 \varphi \sim H^2 f_1^2 D^2 \hat{\delta}^3 , \quad (5.15)$$

and it is apparent that these terms will affect $\tilde{\delta}_2(\mathbf{k})$ and $\tilde{\kappa}_2(\mathbf{k})$ due to their time-dependence, despite being at the third order in perturbations. Note that the quadratic terms in the dynamical equations yield the standard F_2 and G_2 in Fourier space or (3.52) and (3.53) in configuration space.

To implement this change in the perturbative approach in (5.6), we introduce a *time-dependent* third-order SPT kernel

$$F_3^{\delta_2}(t, \mathbf{k}) \equiv \frac{1}{D_1(t)} \tilde{F}_3^{\delta_2}(\mathbf{k}) , \quad (5.16)$$

such that the density fluctuation is

$$\delta(t, \mathbf{k}) \propto D\hat{\delta} + D^2 F_2 \hat{\delta}^2 + D^3 (F_3 + F_3^{\delta_2}) \hat{\delta}^3 = D\hat{\delta} + D^2 (F_2 - \tilde{F}_3^{\delta_2} \Delta \mathcal{R}) \hat{\delta}^2 + D^3 F_3 \hat{\delta}^3 , \quad (5.17)$$

and similarly so for κ and $G_3^{\kappa_2}$. Therefore, the additional terms in the algebraic equations (5.12) are

$$2\tilde{F}_3^{\delta_2} - \tilde{G}_3^{\kappa_2} = \tilde{\mathbb{C}}_1(\mathbf{k}) \equiv \frac{\mathbb{C}_1(t, \mathbf{k})}{H f_1 D^2} \quad \text{and} \quad 5\tilde{G}_3^{\kappa_2} - 3\tilde{F}_3^{\delta_2} = 2\tilde{\mathbb{C}}_2(\mathbf{k}) \equiv \frac{2\mathbb{C}_2(t, \mathbf{k})}{H^2 f_1^2 D^2} , \quad (5.18)$$

where \mathbb{C}_1 and \mathbb{C}_2 represent respectively the third order terms in the relativistic dynamical equations (5.3) and (5.4). The relativistic corrections to the SPT kernels are then

$$\tilde{F}_3^{\delta_2} = \frac{5\tilde{\mathbb{C}}_1 + 2\tilde{\mathbb{C}}_2}{7} \quad \text{and} \quad \tilde{G}_3^{\kappa_2} = \frac{3\tilde{\mathbb{C}}_1 + 4\tilde{\mathbb{C}}_2}{7} . \quad (5.19)$$

Using linear-order perturbation variables

$$\kappa_1^{(1)}(t, \mathbf{k}) = H f_1 D \hat{\delta}(\mathbf{k}) = \frac{k^2}{a^2} \chi_1^{(1)}(t, \mathbf{k}) , \quad \mathbf{v}^{(1)}(t, \mathbf{k}) = i a \frac{\mathbf{k}}{k^2} \kappa_1^{(1)}(t, \mathbf{k}) , \quad \varphi(\mathbf{k}) = \frac{1}{k^2} \hat{\delta}(\mathbf{k}) , \quad (5.20)$$

the third order terms of the relativistic corrections can be computed [17, 18] as

$$\tilde{\mathbb{C}}_1 = -\frac{2\mathbf{q}_2 \cdot \mathbf{q}_3}{q_1^2 q_2^2} - \frac{\mathbf{q}_{12} \cdot \mathbf{q}_3}{q_{12}^2} \left(-\frac{2}{q_1^2} + \frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{q_1^2 q_2^2} - \frac{3}{2} \frac{\mathbf{q}_{12} \cdot \mathbf{q}_2}{q_{12}^2 q_2^2} - \frac{3}{2} \frac{\mathbf{q}_{12} \cdot \mathbf{q}_1}{q_{12}^2 q_1^2} \frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{q_1^2 q_2^2} \right) , \quad (5.21)$$

$$\begin{aligned} \tilde{\mathbb{C}}_2 = & \frac{2}{3} \frac{\mathbf{q}_2 \cdot \mathbf{q}_3}{q_1^2 q_2^2} - 4 \left[\frac{\mathbf{k} \cdot \mathbf{q}_3}{q_3^2} \frac{\mathbf{q}_2 \cdot \mathbf{q}_3}{q_1^2 q_2^2} - \frac{1}{3} \frac{\mathbf{k} \cdot \mathbf{q}_2}{q_1^2 q_2^2} \right] \\ & + \left[\frac{2}{3} + \frac{\mathbf{q}_{12} \cdot \mathbf{q}_3}{q_3^2} \left(1 - \frac{k^2}{q_{12}^2} \right) \right] \left[-\frac{2}{q_1^2} + \frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{q_1^2 q_2^2} - \frac{3}{2} \frac{\mathbf{q}_{12} \cdot \mathbf{q}_2}{q_{12}^2 q_2^2} - \frac{3}{2} \frac{\mathbf{q}_{12} \cdot \mathbf{q}_1}{q_{12}^2 q_1^2} \frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{q_1^2 q_2^2} \right] , \end{aligned} \quad (5.22)$$

where the kernels need to be symmetrized over the arguments. Note that $F_3^{\delta_2} \propto 1/(D_1 k^2) \propto (\mathcal{H}/k)^2$ is dimensionless, as expected.

This derivation of the relativistic corrections, so far, is essentially equivalent to those in [17, 18]. However, as we discussed in Section 4.2, the density fluctuation at the early time t_0 is *not* linear order due to the nonlinearity in the constraint equation (4.15), even if the initial condition \mathcal{R} is a linear-order Gaussian variable and the initial epoch is set $t_0 \rightarrow 0$. To accommodate this intrinsic nonlinearity to the standard Fourier kernels, we need to introduce additional time-dependent kernels:

$$F_2^{\delta_1}(t, \mathbf{k}) \equiv \frac{1}{D_1(t)} \tilde{F}_2^{\delta_1}(\mathbf{k}) \quad \text{and} \quad F_3^{\delta_1}(t, \mathbf{k}) \equiv \frac{1}{D_1^2(t)} \tilde{F}_3^{\delta_1}(\mathbf{k}) , \quad (5.23)$$

where two time-independent kernels are

$$\tilde{F}_2^{\delta_1} = -\frac{1}{k^2} \left[\frac{3}{2} \frac{k^2 \mathbf{q}_1 \cdot \mathbf{q}_2}{q_1^2 q_2^2} + 2 \left(\frac{k^2}{q_1^2} + \frac{k^2}{q_2^2} \right) \right] \quad \text{and} \quad \tilde{F}_3^{\delta_1} = 3 \left(\frac{\mathbf{q}_2 \cdot \mathbf{q}_3}{q_1^2 q_2^2 q_3^2} + \text{cycl.} \right) + 4 \left(\frac{1}{q_1^2 q_2^2} + \text{cycl.} \right) . \quad (5.24)$$

As in (3.29), the higher-order terms in κ_1 are identical to δ_1 , and so are their kernels.

Similarly for δ_2 and κ_2 , this intrinsic nonlinearity of the second order terms in F_2 and G_2 in the Fourier space or (3.52) and (3.53) yields additional third-order terms described in the last lines of (3.57) and (3.58), and this will modify $F_3^{\delta_2}$ and $G_3^{\kappa_2}$:

$$\Delta F_3^{\delta_2}(t, \mathbf{k}) \equiv \frac{1}{D_1(t)} \Delta \tilde{F}_3^{\delta_2}(\mathbf{k}) \quad \text{and} \quad \Delta G_3^{\kappa_2}(t, \mathbf{k}) \equiv \frac{1}{D_1(t)} \Delta \tilde{G}_3^{\kappa_2}(\mathbf{k}) , \quad (5.25)$$

where two time-independent spatial kernels are

$$\begin{aligned}\Delta\tilde{F}_3^{\delta_2}(\mathbf{k}) &= \frac{2^2}{5^2} \left[\frac{10}{7} + \frac{\mathbf{q}_1 \cdot \mathbf{q}_{23}}{q_1^2 q_{23}^2} (q_1^2 + q_{23}^2) + \frac{4}{7} \frac{(\mathbf{q}_1 \cdot \mathbf{q}_{23})^2}{q_1^2 q_{23}^2} \right] \tilde{F}_2^{\delta_1}(\mathbf{q}_2, \mathbf{q}_3), \\ \Delta\tilde{G}_3^{\kappa_2}(\mathbf{k}) &= \frac{2^2}{5^2} \left[\frac{6}{7} + \frac{\mathbf{q}_1 \cdot \mathbf{q}_{23}}{q_1^2 q_{23}^2} (q_1^2 + q_{23}^2) + \frac{8}{7} \frac{(\mathbf{q}_1 \cdot \mathbf{q}_{23})^2}{q_1^2 q_{23}^2} \right] \tilde{F}_2^{\delta_1}(\mathbf{q}_2, \mathbf{q}_3),\end{aligned}\tag{5.26}$$

where the kernels need to be symmetrized over the arguments.

6 Discussions

The proper-time hypersurface of nonrelativistic matter flows is a physically well-defined global time-slicing that a local observer moving with nonrelativistic matter can establish. Galaxy bias in the Newtonian context can be naturally generalized in this proper-time hypersurface in the relativistic context [13]. As the first step toward this direction, we have derived the third-order analytic solutions for the matter density and the velocity fluctuations in the proper-time hypersurface, providing essential ingredients for computing the subtle one-loop corrections to the matter power spectrum.

For the first time, we have derived the *exact* analytic solutions of the matter density and the velocity fluctuations in a Λ CDM universe, accounting for the nonlinear relativistic effects and greatly extending the results of [16] in the EdS universe. Our general approach to solving the nonlinear dynamical equations allows us to derive the solutions in a Λ CDM universe, in which the time-dependence of the solutions is more complicated than that in the EdS universe. In particular, we have derived the explicit solutions to the Green's functions for the growth factors, for which only the differential equations were known in literature. Our solutions are composed of the standard Newtonian solutions and the relativistic corrections. Our Newtonian solutions with the exact time-dependences show that the standard assumption that the solution is separable in its time and spatial dependences is *invalid*, rendering the growth of perturbations scale-dependent in general relativity. However, the recent study in [34] of the Newtonian perturbation theory shows that as long as the linear-order growth factor is properly considered, the errors in the power spectrum with the standard assumption are rather small at $k < 0.2h\text{Mpc}^{-1}$, while it amounts to $\sim 0.5 - 1\%$ at $k \gtrsim 0.2h\text{Mpc}^{-1}$.

On large scales, which is the scale of our interest, the relativistic effects in galaxy clustering become important, providing unique opportunities to probe subtle properties of gravity and the physics relevant for the early universe. For example, the primordial non-Gaussianity can be probed with the galaxy power spectrum via its unique scale-dependence on large scales [35]. As this unique signature of the early universe is also a relativistic effect, we need to take into consideration other relativistic effects in measuring the primordial non-Gaussianity signature [5]. The matter density fluctuation constitutes the dominant contribution to the galaxy clustering measurements on all scales, and we have derived the exact relativistic corrections in a Λ CDM universe to the matter density fluctuation. Previously, the third-order relativistic solutions for the matter density and the velocity fluctuations were derived [18] in the comoving gauge, assuming the EdS universe. For a pressureless medium, the comoving gauge condition corresponds to the proper-time hypersurface [13]. Their solutions agree with ours in the Newtonian part, while there exist differences in the relativistic corrections. The nonlinear constraint equations in general relativity impose nonlinearity in the matter density fluctuation at early time, even with the initial condition set up by the comoving-gauge curvature potential at the linear order in perturbations. We have demonstrated that the difference in the two solutions is exactly due to the initial nonlinearity in the matter density fluctuation, imposed by the ADM energy constraint. Given that the initial condition is set up by inflation at early time, when there is no matter fluid to begin with, our solutions are more appropriate for analyzing the nonlinear growth of the matter density fluctuation in general relativity.

In the era of precision measurements from numerous current and future galaxy surveys, the subtle relativistic effects in galaxy clustering can be utilized to distinguish various inflationary models or competing dark energy models on large scales. The third-order analytic solutions for the matter density fluctuation in this work provide such a first step.

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